Mathematics Claim #2

PROBLEM SOLVING

Students can solve a range of complex well-posed problems in pure and applied mathematics, making productive use of knowledge and problem solving strategies.

Rationale for Claim #2

Assessment items and tasks focused on this claim include well-posed problems in pure mathematics and problems set in context. *Problems* are presented as items and tasks that are well posed (that is, problem formulation is not necessary) and for which a solution path is not immediately obvious.[8] These problems require students to construct their own solution pathway, rather than to follow a provided one. Such problems will therefore be less structured than items and tasks presented under Claim #1, and will require students to select appropriate conceptual and physical tools to use.

At the heart of doing mathematics is making sense of problems and persevering in solving them[9]. This claim addresses the core of mathematical expertise – the set of competences that students can use when they are confronted with challenging tasks.

"Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary." (Practice 1, CCSSM)

Problem solving, which, of course, builds on a foundation of knowledge and procedural proficiency, sits at the core of *doing* mathematics. Proficiency at problem solving requires students to choose to use concepts and procedures from across the content domains and check their work using alternative methods. As problem-solving skills develop, student understanding of and access to mathematical concepts becomes more deeply established.

For example, "older students might, depending on the context of the problem, transform algebraic

[8] Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.
[9] See, e.g., Halmos, P. (1980). The heart of mathematics. *American Mathematical Monthly*, 87, 519-524

expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can approach and solve a problem by drawing upon different mathematical characteristics, such as: correspondences among equations, verbal descriptions of mathematical properties, tables graphs and diagrams of important features and relationships, graphical representations of data, and regularity or irregularity of trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches." (Practice 1, CCSSM)

Development of the capacity to solve problems also corresponds to the development of important meta-cognitive skills such as oversight of a problem-solving process while attending to the details. Mathematically proficient students continually evaluate the reasonableness of their intermediate results, and can step back for an overview and shift perspective. (Practice 7, Practice 8, CCSM)

Problem solving also requires students to identify and select the tools that are necessary to apply to the problem. The development of this capacity – to frame a problem in terms of the steps that need to be completed and to review the appropriateness of various tools that are available – are critical to further learning in mathematics, and generalize to real-life situations. This includes both mathematical tools and physical ones:

"Tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge." (Practice 5, CCSSM)

What sufficient evidence looks like for Claim #2

Although items and tasks designed to provide evidence for this claim must primarily assess the student's ability to identify the problem and to arrive at an acceptable solution, mathematical problems nevertheless require students to apply mathematical concepts and procedures. Thus, though the primary purpose of items/tasks associated with this claim is to assess problem-solving skills, these items/tasks might also contribute to evidence that is gathered for Claim #1.

Properties of items/tasks that assess this claim: The assessment of many relatively discrete and/or single-step problems can be accomplished using short constructed response items, or even computer-enhanced or selected response items.

Additionally, more extensive constructed response items can effectively assess multi-stage problem solving and can also indicate unique and elegant strategies used by some students to solve a given problem, and can illuminate flaws in student's approach to solving a problem. These tasks could:

- Present non-routine[10] problems where a substantial part of the challenge is in deciding what to do, and which mathematical tools to use; and
- Involve chains of autonomous[11] reasoning, in which some tasks may take a successful student 5 to 10 minutes, depending on the age of student and complexity of the task.

A distinctive feature of both single-step and multi-step items and tasks for Claim #2 is that they are "well-posed". That is, whether the problem deals with pure or applied contexts, the problem itself is completely formulated; the challenge is in identifying or using an appropriate solution path. Two examples of well-posed problems are provided below, following the Assessment Targets for Claim #2.

Because problems like these might be new to many students, especially on a state-level assessment, it will be worthwhile to explore developing scaffolded supports within the assessment to facilitate entry and assess student progress towards expertise. The degree of scaffolding for individual students could be determined as part of the adaptability of the computer-administered test. Even for such "scaffolded tasks," part of the task will involve a chain of autonomous reasoning. Additionally, because some multi-stage problem-solving tasks might present significant cognitive complexity, consideration should be given to framing more complex problem solving tasks with mathematical concepts and procedures that have been mastered in an earlier grade.

Problems in pure mathematics: These are well-posed problems within mathematics where the student must find an approach, choose which mathematical tools to use, carry the solution through, and explain the results.

Design problems: These problems have much the same properties but within a design context from the real, or a fantasy, world. See, for example, "sports bag" from the assessment sampler.

Planning problems: Planning problems involve the coordinated analysis of time, space, cost – and people. They are design tasks with a time dimension added. Well-posed problems of this kind assess the student's ability to make the connections needed between different parts of mathematics.

[10] As noted earlier, by "non-routine" we mean that the student will not have been taught a closely similar problem, so will not be expected to *remember* a solution path but will have to *adapt* or *extend* their earlier knowledge to find one.
[11] By "autonomous" we mean that the student responds to a single prompt, without further guidance within the task.

This is not a complete list; other types of tasks that fit the criteria above may well be included. But a balanced mixture of these types will provide enough evidence for Claim #2, as well as contributing evidence with regard to Claim #1. Illustrative examples of each type are shown in the item and task specifications as well as in the publicly available practice tests available online.

Scoring rubrics for extended response items and tasks should be consistent with the expectations of this claim, giving substantial credit to the choice of appropriate methods of tackling the problem, to reliable skills in carrying it through, and to explanations of what has been found. Scoring for Claims 2, 3, & 4 is anchored to the general rubrics.

Accessibility and Claim #2: This claim about mathematical problem solving focuses on the student's ability to make sense of problems, construct pathways to solving them, persevering in solving them, and the selection and use of appropriate tools. This claim includes student use of appropriate tools for solving mathematical problems, which for some students may extend to tools that provide full access to the item or task and to the development of reasonable solutions. For example, students who are blind and use Braille or assistive technology such as text readers to access written materials, may demonstrate their modeling of physical objects with geometric shapes using alternate formats. Students who have physical disabilities that preclude movement of arms and hands should not be precluded from demonstrating with assistive technology their use of tools for constructing shapes. As with Claim #1, access via text to speech and expression via scribe, computer, or speech to text technology will be important avenues for enabling many students with disabilities to show what they know and can do in relation to framing and solving complex mathematical problems.

With respect to English learners, the expectation for verbal explanations of problems will be more achievable if formative materials and interim assessments provide illustrative examples of the communication required for this claim, so that ELL students have a better understanding of what they are required to do. In addition, formative tools can help teachers teach ELL students ways to communicate their ideas through simple language structures in different language modalities such as speaking and writing. Finally, attention to English proficiency in shaping the delivery of items (e.g. native language or linguistically modified, where appropriate) and the expectations for scoring will be important.

Assessment Targets for Claim #2

Claim #2 is aligned to the mathematical practices from the MCCSS. For this reason, the Assessment Targets are all *acts of problem solving* that are consistent across grades and also evolve across grades. Consistent with the above discussion, these acts of problem solving are also tied to content (CCSSM, p. 8).

SUMMATIVE ASSESSMENT TARGETS **Providing Evidence Supporting Claim #2**
Claim #2: Students can solve a range of complex well-posed problems in pure and applied mathematics, making productive use of knowledge and problem-solving strategies.
To preserve the focus and coherence of the standards as a whole, tasks must draw clearly on knowledge and skills that are articulated in the content standards. At each grade level, the content standards offer natural and productive settings for generating evidence for Claim #2. These connections are specified below. Tasks generating evidence for Claim #2 in a given grade will draw upon knowledge and skills articulated in the progression of standards up through that grade, though more complex problem-solving tasks may draw upon knowledge and skills from lower grade levels. Any given task will provide evidence for several of the following assessment targets. Each of the following targets should not lead to a separate task: it is in *using* content from different areas, including work studied in earlier grades, that students demonstrate their problem-solving proficiency.
Content clusters and domains recommended for the majority of Claim 2 item development are given below. Tasks can center on a single cluster or standard listed, or synthesize across listed clusters or standards.

Grade 3	**Grade 4**	**Grade 5**	**Grade 6**	**Grade 7**	**Grade 8**	**HS**
3.OA.A	4.OA.A	5.NBT.B	6.RP.A	7.RP.A	8.EE.B	N-Q.A
3.OA.D	4.NBT.B	5.NF.A	6.NS.A	7.NS.A	8.EE.C	A-SSE.A
3.NBT.A*	4.NF.A	5.NF.B	6.NS.C	7.EE.A	8.F.A	A-SSE.B
3.MD.A	4.NF.B	5.MD.A*	6.EE.A	7.EE.B	8.F.B*	A-CED.A
3.MD.B*	4.NF.C	5.MD.C	6.EE.B	7.G.A*	8.G.A	A-REI.2
3.MD.C	4.MD.A*	5.G.A*	6.EE.C	7.G.B*	8.G.B	A-REI.B
3.MD.D*	4.MD.C*		6.G.A*		8.G.C*	A-REI.C
						A-REI.D
						F-IF.A
						F-IF.B
						F-IF.C
						F-BF.A
						G-SRT.C
						S-ID.C
						S-CP.A

* Denotes additional and supporting clusters

Target A: Apply mathematics to solve well-posed problems in pure mathematics and those arising in everyday life, society, and the workplace. (DOK 2, 3)

Target B: Select and use appropriate tools strategically. (DOK 1, 2)

Target C: Interpret results in the context of a situation. (DOK 2)

Target D: Identify important quantities in a practical situation and map their relationships (e.g., using diagrams, two-way tables, graphs, flowcharts, or formulas). (DOK 1, 2, 3)

Example of a short answer task for Claim #2

"Toys for Charity" **(First-year Algebra)**

Phil and Cathy want to raise money for charity. They decide to make and sell wooden toys.
They could make them in two sizes: small and large.

Phil will carve them from wood. A small toy takes 2 hours to carve and a large toy takes 3 hours to carve.
Phil only has a total of 24 hours available for carving.

Cath will decorate them. She only has time to decorate 10 toys.

The small toy will make $8 for charity.
The large toy will make $10 for charity.

They want to make as much money for charity as they can.

How many small and large toys should they make?

How much money will they then make for charity?

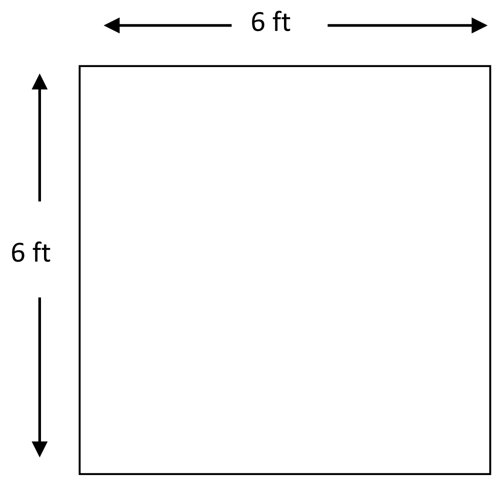
For the above example, scaffolding could prompt the student to think about questions like:

1. If they were to make only small toys, how much money would they make for charity?
2. If they were to make 2 small toys, how many large ones could they also make?

Example of an extended response task for Claim #2

Making a Water Tank **(Grade 11 – students provided graphing calculator as a tool)**

A square metal sheet (6 feet x 6 feet) is to be made into an open-topped water tank by cutting squares from the four corners of the sheet, and bending the four remaining rectangular pieces up, to form the sides of the tank. These edges will then be welded together.



A. How will the final volume of the tank depend upon the size of the squares cut from the corners?

Describe your answer by:

i) Sketching a rough graph

ii) explaining the shape of your graph in words

iii) writing an algebraic formula for the volume

B. How large should the four corners be cut, so that the resulting volume of the tank is as large as possible?